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An Approach to Target Tracking

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ABSTRACT

This report presents an approach to point target tracking based on sequential filtering techniques. The tracking problem is defined in terms of a nonlinear vector differential equation and an appropriate state vector. A Bayesian formulation for the problem is selected which results in a least-squares filter solution. Linearization techniques essential to this approach are incorporated into the development of the solution. A computer program which implements the complete solution algorithm is presented. As part of this computer realization, numerical integration of the equations of motion and numerical evolution of the estimate covariance matrix are discussed in detail.

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CONTENTS

AN APPROACH TO TARGET TRACKING

I. INTRODUCTION

In the past several years a number of publications in the control theory literature has dealt with the problem of estimating state variables associated with nonlinear systems. Much of this work is based on the early linear estimation techniques established by Kalman and Bucy.[1,2]

In an engagement between a missile attack and a defense system, one of the crucial modes of the defense system will be point target tracking. As will be shown, the problem of tracking a point target with a radar is a nonlinear parameter estimation problem. The purpose of this report is to apply the techniques of modern control theory to that problem. A variety of topics essential to understanding the philosophy of nonlinear, recursive estimation techniques is presented, with application to a specific re-entry tracking problem.

The problem considers a missile trajectory which may be described by the nonlinear vector differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{1-1}$$

where $\underline{x} = \underline{x}(t)$ is an n-dimensional vector whose components define the trajectory of the approaching missile. Observations on this target are available in the form

$$\underline{y}(t_k) = \underline{y}_k = \underline{h}(\underline{x}_k) + \underline{v}_k \tag{1-2}$$

where $\underline{h}(\underline{x})$ is an m-dimensional vector function and $\underline{v}_k$ is Gaussian white noise. The estimator $\hat{\underline{x}}_k$ of $\underline{x}_k$ will be required to utilize the observation $\underline{y}_k$ along with the information inherent in $\underline{f}$ and $\underline{h}$ and a previous estimate $\hat{\underline{x}}_{k-1}$ to obtain the best (in some sense) estimate. In particular, we will give explicit equations for the problem and obtain a solution for the case of a seven-dimensional state vector with three position components, three velocity components, and the drag-to-weight ratio. The observation $\underline{y}_k$ is four dimensional with range, azimuth, elevation, and range rate as components.

Each of the following five sections presents a particular facet of nonlinear estimation problems. Section II presents the general linearization techniques which are used to modify the nonlinear expressions and establish approximate expressions amenable to computer processing. Section III considers the specific problem of obtaining an estimator and its covariance. Section IV briefly discusses the actual computer algorithms used to estimate the state variables. Section V tabulates the actual state vectors and matrices defined for the ballistic missile re-entry problem. Section VI treats in detail the techniques used in the numerical integration of the nonlinear system of equations (1-1). Also described in Sec. VI is an incremental linearization

technique used to update the estimate covariance matrices. It is this particular area of covariance estimation which is least developed in the field of nonlinear estimation problems. Much work remains to be done before all phenomena involved in such estimation problems can be fully understood.

II. LINEARIZATION ABOUT A TRAJECTORY IN NONLINEAR SYSTEMS

In the general nonlinear estimation problem we are confronted with the set of equations (1-1) and (1-2). Unfortunately, the ability to obtain closed-form solutions to such problems is severely restricted by the techniques presently available. As a rule, we must resort to numerical methods and computer techniques in order to achieve satisfactory solutions to such problems. This section presents some of the linearization techniques useful in preparing nonlinear problems for computer solution.

Let the system under consideration be described by the vector differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{2-1}$$

where

$\underline{x}$ = n-dimensional state vector

$\underline{f}$ = n-dimensional vector function of $\underline{x}$.

It will be assumed that $\underline{f}$ satisfies conditions for unique solutions to (2-1) to exist. A solution starting at time t_o and initial state $\underline{x}_o$ will be denoted by

$$\underline{x}^o(t) = \underline{x}(t, t_o, \underline{x}_o) \quad . \tag{2-2}$$

Equation (2-1) can be rewritten in the form of a nonlinear integral equation

$$\underline{x}^o(t) = \underline{x}_o + \int_{t_o}^{t} \underline{f}\,[\underline{x}(\tau)]\, d\tau \tag{2-3}$$

which is more convenient when solutions are obtained by numerical integration, as is the case here. In the subsequent discussion it will be assumed that such a solution corresponding to $\underline{x}_o$, t_o has been found, i.e., $\underline{x}^o(t)$ is known. (An algorithm for the numerical solution of (2-1) is given in Sec. VI.)

Algorithms for the estimation of state variables of dynamic systems from noisy data (observations) require that the covariance matrix of $\underline{x}^o(t)$ be found, given the covariance matrix of $\underline{x}_o$. For nonlinear systems this is usually not possible directly, so that linearizations have to be introduced. Thus the equation of motion (2-1) is linearized about the nominal trajectory $\underline{x}^o(t)$. Consider the effect a small change in $\underline{x}_o$ has on $\underline{x}^o(t)$.

Let

$$\underline{x}_o^* = \underline{x}_o + \delta\underline{x}_o$$

and

$$x^*(t) = \underline{x}^o(t) + \delta\underline{x}(t) = \underline{x}(t, t_o, \underline{x}_o^*) = \underline{x}(t, t_o, \underline{x}_o + \delta\underline{x}_o) \quad .$$

Then, assuming that $\partial\underline{f}/\partial\underline{x}$ exists,

$$\dot{\underline{x}}^*(t) = \underline{f}\,[\underline{x}^*(t)] = \underline{f}\,[\underline{x}^o(t)] + \left.\frac{\partial \underline{f}(x)}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}^o(t)} \delta\underline{x}(t) + \text{higher order terms}$$

or, since $\dot{\underline{x}}^*(t) = \dot{\underline{x}}^o(t) + \delta\dot{\underline{x}}(t)$,

$$\delta\dot{\underline{x}}(t) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}^o(t)} \delta\underline{x}(t) + \text{higher order terms} \quad . \tag{2-4}$$

Since it was assumed that $\underline{x}^o(t)$ is known, (2-4) can be rewritten as

$$\delta\dot{\underline{x}}(t) = \underline{A}(t)\,\delta\underline{x} + \text{higher order terms} \tag{2-5}$$

where

$$\underline{A}(t) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}^o(t)} \quad .$$

Now for small enough $\delta\underline{x}$, the higher order terms can be neglected in (2-5) so that

$$\delta\dot{\underline{x}}(t) = \underline{A}(t)\,\delta\underline{x}(t) \quad . \tag{2-6}$$

By restricting $\delta\underline{x}_o$ to be small enough, $\delta\underline{x}$ [and hence $\underline{x}^*(t)$] can be approximated arbitrarily close to its true value by (2-6). The basic assumption made in nonlinear estimation problems can now be stated. If $\underline{x}^*(t)$ denotes the true state of the system and $\underline{x}^o(t)$ the estimate at time t, based on the estimate $\underline{x}_o$ at time t_o, then

$$\begin{aligned}\underline{P}(t) &= E\{[\underline{x}^*(t) - \underline{x}^o(t)]\,[\underline{x}^*(t) - \underline{x}^o(t)]'\} \\ &= E\{[\delta\underline{x}(t)]\,[\delta\underline{x}(t)]'\} \quad .\end{aligned} \tag{2-7}$$

If the error in the estimate $\underline{x}^o(t)$ is sufficiently small, we can use (2-6) to evaluate $\delta\underline{x}(t)$. Letting the transition matrix associated with (2-6) be denoted by $\phi(t, t_o)$, $\delta\underline{x}(t)$ is given by

$$\delta\underline{x}(t) = \phi(t, t_o)\,\delta\underline{x}_o \quad . \tag{2-8}$$

Hence

$$\begin{aligned}\underline{P}(t) &= E\{[\delta\underline{x}(t)]\,[\delta\underline{x}(t)]'\} \\ &= E\{[\phi(t, t_o)\,\delta\underline{x}_o]\,[\phi(t, t_o)\,\delta\underline{x}_o]'\} \\ &= \phi(t, t_o)\,E(\delta\underline{x}_o\delta\underline{x}_o')\,\phi'(t, t_o)\end{aligned}$$

or

$$\underline{P}(t) = \phi(t, t_o)\,\underline{P}(t_o)\,\phi'(t, t_o) \quad . \tag{2-9}$$

The same result is found by obtaining the differential equation for $\phi(t, t_o)$ directly. To do this, note that from (2-8) it follows that

$$\phi(t, t_o) = \frac{\partial \underline{x}(t)}{\partial \underline{x}_o} \quad . \tag{2-10}$$

Therefore, taking the partial derivative indicated in (2-10) in (2-1) yields

$$\frac{\partial}{\partial \underline{x}_o}\left(\frac{d\underline{x}}{dt}\right) = \frac{\partial}{\partial \underline{x}_o}\left[\underline{f}(\underline{x})\right]$$

or

$$\frac{d}{dt}\left(\frac{\partial \underline{x}}{\partial \underline{x}_o}\right) = \frac{\partial \underline{f}(\underline{x})}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \underline{x}_o} \tag{2-11}$$

or rewriting (2-11) using (2-10) yields

$$\frac{d}{dt}\phi(t, t_o) = \frac{\partial \underline{f}(\underline{x})}{\partial \underline{x}_o}\phi(t, t_o) \quad . \tag{2-12}$$

Since $\phi(t, t_o)$ is to be evaluated along the nominal trajectory $\underline{x}^o(t)$, $[\partial \underline{f}(\underline{x})/\partial \underline{x}_o]$ has to be evaluated along that same trajectory. Hence, using the notation of (2-5),

$$\frac{d}{dt}\phi(t, t_o) = \underline{A}(t)\,\phi(t, t_o) \quad . \tag{2-13}$$

Equations (2-13) and (2-6) are equivalent in that for linear systems, the transition matrix satisfies its own differential equation. From (2-13) it is not apparent where approximations were introduced, since it was not necessary anywhere to neglect higher order terms as it was for (2-6). However, it is quite easy to see that in order to use the solution of (2-13) to evaluate the covariance matrix $\underline{P}(t)$, the assumption that $\delta\underline{x}(t)$ is small has to be made again. For in that case, it follows directly that

$$\delta\underline{x} = \phi(t, t_o)\,\delta\underline{x}_o$$

and hence

$$\underline{P}(t) = \phi(t, t_o)\,\underline{P}(t_o)\,\phi'(t, t_o) \quad . \tag{2-14}$$

III. DERIVATION OF THE TRACKING EQUATIONS

In Sec. II, discussion centered about a system which could be described by a nonlinear vector differential equation (2-1). In this section, it will be assumed that an exact solution to such a nonlinear system of equations is available and can be expressed in the discrete form

$$\underline{x}_k = \underline{g}(\underline{x}_{k-1}) + \underline{u}_k \quad . \tag{3-1}$$

An expression relating these state variables $\underline{x}_k$ to the observations may be written as

$$\underline{y}_k = \underline{h}(\underline{x}_k) + \underline{v}_k \quad . \tag{3-2}$$

A separate report will discuss some implications of the transition from the continuous to the discrete domain.

It is now possible to derive an estimator for $\underline{x}_k$ utilizing the observations $\underline{y}_k$ if the following assumptions are satisfied:

$\underline{u}_k$, $\underline{v}_k$ = white, zero mean, Gaussian random variables with

$$E(\underline{u}_k\underline{u}_k^T) = \underline{Q}_k$$

$$E(\underline{v}_k\underline{v}_k^T) = \underline{R}_k$$

and $\underline{g}$, $\underline{h}$, and the noise $\underline{u}_k$ and $\underline{v}_k$ are such that for incremental changes, the equations

$$\delta \underline{x}_k \cong \left. \frac{\partial \underline{g}(\underline{x})}{\partial \underline{x}} \right|_{\underline{x}=\underline{x}_{k-1}} \delta \underline{x}_{k-1}$$

$$\delta \underline{y}_k \cong \left. \frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} \right|_{\underline{x}=\underline{x}_k} \delta \underline{x}_k$$

hold. This in effect assumes that the $\underline{x}$ process is Gaussian with a mean whose evolution in time is given by a nonlinear equation. Under these assumptions

$$p(\underline{y}_k | \underline{x}_k) = \frac{1}{(2\pi)^{m/2} \, |\underline{R}_k|^{1/2}} \exp\{-\frac{1}{2} [\underline{y}_k - \underline{h}(\underline{x}_k)]^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\underline{x}_k)]\} \tag{3-3}$$

$$p(\underline{x}_k | Y_{k-1}) \cong \frac{1}{(2\pi)^{n/2} \, |\underline{S}_{k,\,k-1}|^{1/2}} \exp[-\frac{1}{2} (\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})^T$$

$$\times \underline{S}_{k,\,k-1}^{-1} (\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})] \tag{3-4}$$

$$p(\underline{y}_k | Y_{k-1}) \cong \frac{1}{(2\pi)^{m/2} \, |\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k|^{1/2}} \exp\{-\frac{1}{2} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]^T$$

$$\times (\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k)^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]\} \tag{3-5}$$

where

n = dimension of $\underline{x}$

m = dimension of the observation vector $\underline{y}_k$

Y_{k-1} = set of all observations $\underline{y}_j$, $j = 1, 2, \ldots k-1$

$\hat{\underline{x}}_{k,\,k-1} = \underline{g}(\hat{\underline{x}}_{k-1})$

$\hat{\underline{x}}_{k-1}$ = state estimate at time t_{k-1} based on all observations $\underline{y}_k$, $j = 1, 2, \ldots k-1$ (i.e., Y_{k-1})

$\underline{S}_{k,\,k-1} = E\,[\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})^T]$

$$\underline{H}_k = \left. \frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} \right|_{\underline{x}=\hat{\underline{x}}_{k,\,k-1}} .$$

Following Ho and Lee,[3] $p(\underline{x}_k/Y_k)$ is given by

$$p(\underline{x}_k | Y_k) = \frac{p(\underline{x}_k | Y_{k-1})}{p(\underline{y}_k | Y_{k-1})} \, p(\underline{y}_k | \underline{x}_k) \tag{3-6}$$

or

$$p(\underline{x}_k | Y_k) \cong \frac{|\underline{H}_k \underline{S}_{k,k-1} \underline{H}_k^T + \underline{R}_k|^{1/2}}{(2\pi)^{n/2} |\underline{R}_k|^{1/2} |\underline{S}_{k,k-1}|^{1/2}} \exp\{-\frac{1}{2} \{[\underline{y}_k - \underline{h}(\underline{x}_k)]^T \underline{R}_k^{-1}$$

$$\times [\underline{y}_k - \underline{h}(\underline{x}_k)] + (\underline{x}_k - \hat{\underline{x}}_{k,k-1})^T \underline{S}_{k,k-1}^{-1} (\underline{x}_k - \hat{\underline{x}}_{k,k-1})$$

$$- [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]^T (\underline{H}_k \underline{S}_{k,k-1} \underline{H}_k^T + \underline{R}_k)^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]\}\} \quad . \tag{3-7}$$

Because $\underline{x}_k$ occurs nonlinearly in (3-7), it is in general not possible to solve directly for $\hat{\underline{x}}_k$, the conditional expectation of $\underline{x}_k$. However, assuming that the true state $\underline{x}_k$ is close to $\hat{\underline{x}}_{k,k-1}$ and assuming appropriate properties of $\underline{h}(\underline{x})$, the term $\underline{h}(\underline{x}_k)$ can be expanded about $\hat{\underline{x}}_{k,k-1}$ in the form

$$\underline{h}(\underline{x}_k) \cong \underline{h}(\hat{\underline{x}}_{k,k-1}) + \left.\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}}\right|_{\underline{x}=\hat{\underline{x}}_{k,k-1}} (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) \tag{3-8}$$

i.e., only up to first order terms of $\underline{h}(\underline{x}_k)$ are retained in its Taylor series. This implies that in the quadratic form

$$[\underline{y}_k - \underline{h}(\underline{x}_k)]^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\underline{x}_k)] \tag{3-9}$$

up to second order terms should be retained.

Expanding the term $\underline{y}_k - \underline{h}(\underline{x}_k)$ up to second order yields

$$\underline{y}_k - \underline{h}(\underline{x}_k) \cong \underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1}) - \underline{H}_k(\underline{x}_k - \hat{\underline{x}}_{k,k-1})$$

$$- \frac{1}{2} \left\{ \frac{\partial}{\partial \underline{x}} \left[\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) \right] \Bigg|_{\underline{x}=\hat{\underline{x}}_{k,k-1}} \right\} (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) \quad . \tag{3-10}$$

Using (3-10) and retaining all terms up to second order in $\underline{x}_k - \hat{\underline{x}}_{k,k-1}$ yields for (3-9)

$$[\underline{y}_k - \underline{h}(\underline{x}_k)]^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\underline{x}_k)] \cong [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]$$

$$+ (\underline{x}_k - \hat{\underline{x}}_{k,k-1})^T \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) - 2 [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]^T$$

$$\times \underline{R}_k^{-1} \underline{H}_k (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) - \frac{1}{2} (\underline{x}_k - \hat{\underline{x}}_{k,k-1})^T$$

$$\times \left\{ \frac{\partial}{\partial \underline{x}} \left[\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) \right] \Bigg|_{\hat{\underline{x}}_{k,k-1}} \right\}^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]$$

$$- \frac{1}{2} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,k-1})]^T \underline{R}_k^{-1} \left\{ \frac{\partial}{\partial \underline{x}} \left[\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) \right] \Bigg|_{\hat{\underline{x}}_{k,k-1}} \right\}$$

$$\times (\underline{x}_k - \hat{\underline{x}}_{k,k-1}) \quad . \tag{3-11}$$

By writing the last two expressions in (3-11) in terms of the components of the various vectors, it can be shown that they are equivalent to

$$-(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})^T \left(\frac{\partial}{\partial \underline{x}} \left\{ \left[\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} \right]^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] \right\} \Bigg|_{\hat{\underline{x}}_{k,\,k-1}} \right) (\underline{x}_k - \hat{\underline{x}}_{k,\,k-1}) \quad . \tag{3-12}$$

To simplify the subsequent analysis, define

$$\underline{B}_k \triangleq \frac{\partial}{\partial \underline{x}} \left\{ \left[\frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}} \right]^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] \right\} \Bigg|_{\hat{\underline{x}}_{k,\,k-1}} \quad . \tag{3-13}$$

As long as $(\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k - \underline{B}_k)$ is positive definite, (3-7) can be rewritten as

$$\begin{aligned} p(\underline{x}_k | Y_k) \cong \alpha \exp [-\tfrac{1}{2} \{ &(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})^T (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k - \underline{B}_k)(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1}) \\ &- 2(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})^T \underline{H}_k^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] + [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]^T \\ &\times [\underline{R}_k^{-1} - (\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k)^{-1}] [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] \}] \quad . \end{aligned} \tag{3-14}$$

Consider the term

$$\underline{R}_k^{-1} - (\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k)^{-1} = (\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k)^{-1} \underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T \underline{R}_k^{-1} \tag{3-15}$$

which can be rewritten as

$$\begin{aligned} &(\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k)^{-1} \underline{H}_k \underline{S}_{k,\,k-1} (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k) \\ &\qquad \times (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k)^{-1} \underline{H}_k^T \underline{R}_k^{-1} \quad . \end{aligned} \tag{3-15a}$$

After some matrix manipulation, (3-15a) simplifies to

$$\underline{R}_k^{-1} - (\underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k)^{-1} = \underline{R}_k^{-1} \underline{H}_k (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k)^{-1} \underline{H}_k^T \underline{R}_k^{-1} \quad . \tag{3-16}$$

Define

$$\underline{S}_k^{-1} \triangleq (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k - \underline{B}_k) \quad .$$

Add and subtract from the exponent in (3-14) the term

$$[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]^T \underline{R}_k^{-1} \underline{H}_k (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1} \underline{H}_k)^{-1} \underline{B}_k \underline{S}_k \underline{H}_k^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] \quad . \tag{3-17}$$

Then the quadratic term in $\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})$ becomes

$$\begin{aligned} &[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]^T \underline{R}_k^{-1} \underline{H}_k (\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T \underline{R}_k^{-1})^{-1} (\underline{I} + \underline{B}_k \underline{S}_k) \\ &\qquad \times \underline{H}_k^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] - \text{Eq. (3-17)} \quad . \end{aligned}$$

After simplifying, this reduces to

$$[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]^T \underline{R}_k^{-1} \underline{H}_k \underline{S}_k \underline{H}_k^T \underline{R}_k^{-1} [\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] \quad . \tag{3-18}$$

Since $\underline{B}_k$ is proportional to $\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]$, the term (3-17) is of third order in $\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]$. Furthermore, every term in the expansion of (3-17) contains second order derivatives of the components of $\underline{h}(\underline{x})$, so that in (3-18), the term given by (3-17) can be neglected. With this assumption the exponent in (3-14) can be written as

$$\text{exponent} \cong -\frac{1}{2}\{(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})^T \underline{S}_k^{-1}(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})$$

$$-2(\underline{x}_k - \hat{\underline{x}}_{k,\,k-1})\,\underline{H}_k^T\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]$$

$$+[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]^T\,\underline{R}_k^{-1}\underline{H}_k\underline{S}_k\underline{H}_k^T\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]\} \quad . \tag{3-19}$$

By multiplying out the quadratic form below, we can verify that (3-19) can be written as

$$\text{exponent} \cong -\frac{1}{2}[(\underline{x}_k - \{\hat{\underline{x}}_{k,\,k-1} + \underline{S}_k\underline{H}_k^T\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]\})^T\,\underline{S}_k^{-1}$$

$$\times(\underline{x}_k - \{\hat{\underline{x}}_{k,\,k-1} + \underline{S}_k\underline{H}_k^T\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]\})] \quad . \tag{3-20}$$

From (3-20) it follows that the estimate $\hat{\underline{x}}_k$ (the conditional mean) is given by

$$\hat{\underline{x}}_k = \hat{\underline{x}}_{k,\,k-1} + \underline{S}_k\underline{H}_k^T\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]$$

and the covariance of $\hat{\underline{x}}_k$ is given by

$$\underline{S}_k = E[(\underline{x}_k - \hat{\underline{x}}_k)(\underline{x}_k - \hat{\underline{x}}_k)^T] \quad . \tag{3-21}$$

Equation (3-21) can be rewritten

$$\underline{S}_k = \left(\underline{S}_{k,\,k-1}^{-1} + \underline{H}_k^T\underline{R}_k^{-1}\underline{H}_k - \frac{\partial}{\partial\underline{x}}\left\{\left[\frac{\partial\underline{h}(\underline{x})}{\partial\underline{x}}\right]^T \underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})]\right\}\Bigg|_{\hat{\underline{x}}_{k,\,k-1}}\right)^{-1} \tag{3-22}$$

or, using the definition of $\underline{H}_k$,

$$\underline{S}_k = \left(\underline{S}_{k,\,k-1}^{-1} - \frac{\partial}{\partial\underline{x}}\left\{\left[\frac{\partial\underline{h}(\underline{x})}{\partial\underline{x}}\right]^T [\underline{y}_k - \underline{h}(\underline{x})]\right\}\Bigg|_{\hat{\underline{x}}_{k,\,k-1}}\right)^{-1}$$

and $\hat{\underline{x}}_k$ is given by

$$\hat{\underline{x}}_k = \hat{\underline{x}}_{k,\,k-1} + \underline{S}_k\underline{H}_k^T\underline{R}_k^{-1}[\underline{y}_k - \underline{h}(\hat{\underline{x}}_{k,\,k-1})] \quad . \tag{3-23}$$

To determine $\underline{S}_{k,\,k-1}$, expand (3-1) about $\hat{\underline{x}}_{k-1}$ to yield

$$\delta\hat{\underline{x}}_k \cong \frac{\partial\underline{g}}{\partial\underline{x}}\bigg|_{\hat{\underline{x}}_{k-1}} \delta\hat{\underline{x}}_{k-1} + \underline{u}_k \tag{3-24}$$

where

$$\delta\hat{\underline{x}}_{k-1} \triangleq \underline{x}_{k-1} - \hat{\underline{x}}_{k-1}$$

$$\delta\hat{\underline{x}}_k \triangleq \underline{x}_k - \underline{g}(\hat{\underline{x}}_{k-1}) \quad .$$

Keeping only up to first order terms in the expansion of $\delta\hat{\underline{x}}_k$ yields

$$E(\delta\hat{\underline{x}}_k \delta\hat{\underline{x}}_k^T) \overset{\Delta}{=} \underline{S}_{k,\,k-1} = \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\hat{\underline{x}}_{k-1}} \underline{S}_{k-1} \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|^T_{\hat{\underline{x}}_{k-1}} + \underline{Q} \tag{3-25}$$

where we use the fact that $\underline{u}_k$ is independent of $\delta\underline{x}_{k-1}$ and the mean of $\underline{u}_k$ is zero. By similar reasoning, it is easily established that

$$\mathrm{Cov}\,(\underline{y}_k | Y_{k-1}) = \underline{H}_k \underline{S}_{k,\,k-1} \underline{H}_k^T + \underline{R}_k \quad . \tag{3-26}$$

This completes the derivation. It will be noted that, analogous to the continuous case, the covariance matrix given by (3-21) contains the additional term $\underline{B}_k$ [defined by (2-13)] which does not occur if linear filtering theory is applied to the linearized system. For this report, $\underline{u}_k = 0$ has been assumed in later sections.

IV. COMPUTER REALIZATION OF THE ESTIMATOR

A computer realization for (3-23) is now desirable. Since (3-1) is not readily available, part of the computational algorithm must be devoted to obtaining a solution to (2-1). A detailed discussion of this specific problem is presented in Sec. VI. The present section will briefly describe the complete program for obtaining the state variable estimate.

The algorithm used is in fact a valid result for three different approaches to the estimation problem -- that of Sec. III and those of Mowery[4] and Cox.[5] The similarity in the solutions for all three cases is due to the fact that the assumptions made in each case reduce the problem in the final stages to minimizing a quadratic form involving the observations, predicted values of the state vector, and functions of that predicted state. Also of interest is the flexibility of programming the solution in two different ways (see part C of this section).

The equations describing the system are given by

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \qquad \text{[Eq. (2-1)]}$$

$$\underline{y}_k \overset{\Delta}{=} \underline{y}(t_k) = \underline{h}(\underline{x}_k) + \underline{v}_k \qquad \text{[Eq. (3-2)]}$$

where

$\underline{x}$ = state vector

$\underline{y}_k$ = observation vector at time t_k

$\underline{x}_k$ = state vector at time t_k

$\underline{f}, \underline{h}$ = vector valued functions

$\underline{v}_k$ = white, Gaussian, zero mean noise .

In Fig. 1, an overall block diagram is given for the routine. Figures 2 and 3 show the detailed flow diagrams for the problem. The following definitions have been used:

$\hat{\underline{x}}_{k+1,\,k}$ = state estimate (prediction of state) at time t_{k+1}, based only on the estimate at time t_k

$\hat{\underline{x}}_{k+1}$ = state estimate at time t_{k+1}, based on $\underline{y}_{k+1}$ and $\hat{\underline{x}}_{k+1,\,k}$

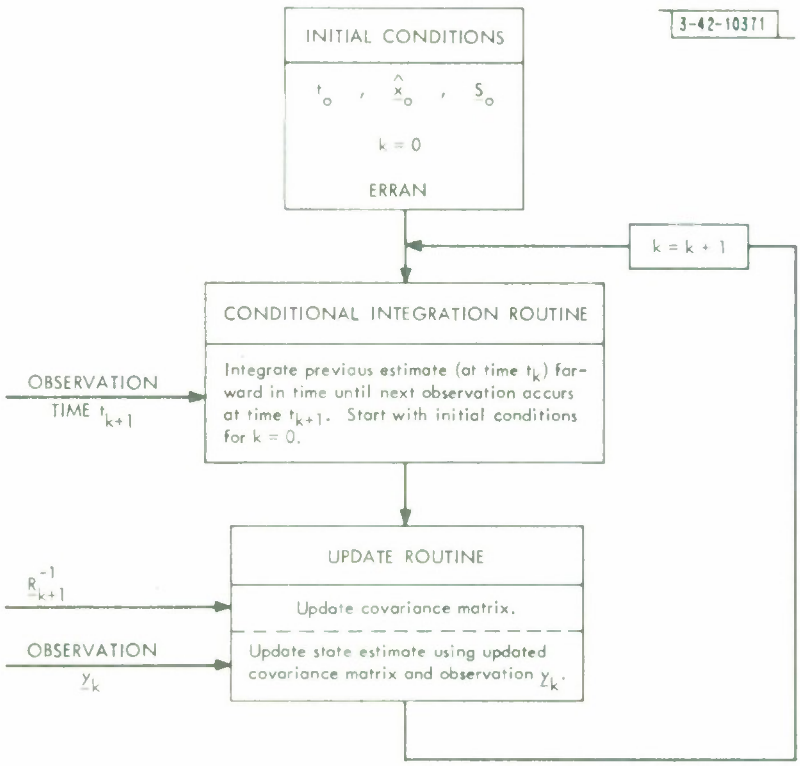

Fig. 1. Estimation routine.

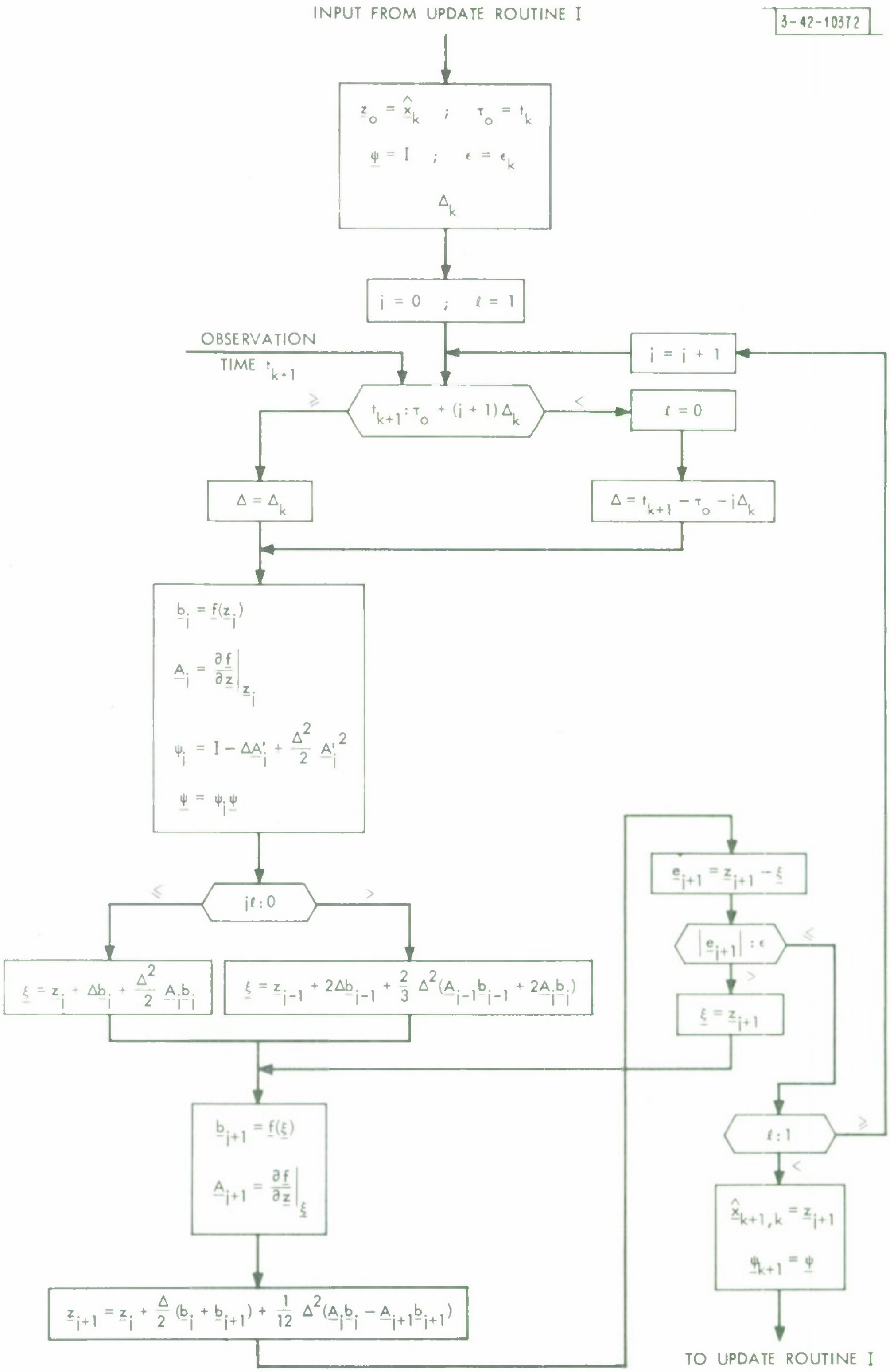

Fig. 2(a). Conditional integration routine I.

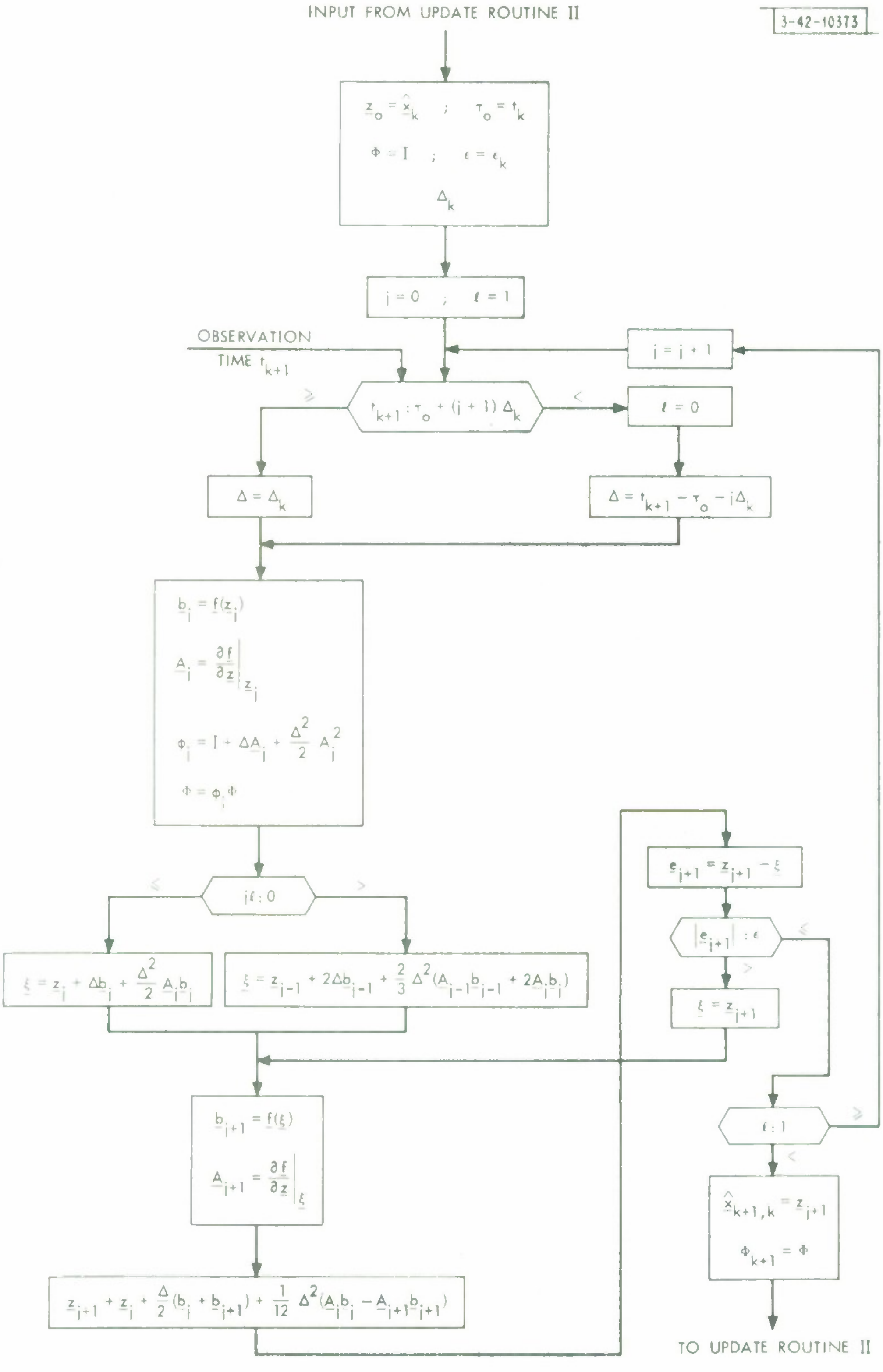

Fig. 2(b). Conditional integration routine II.

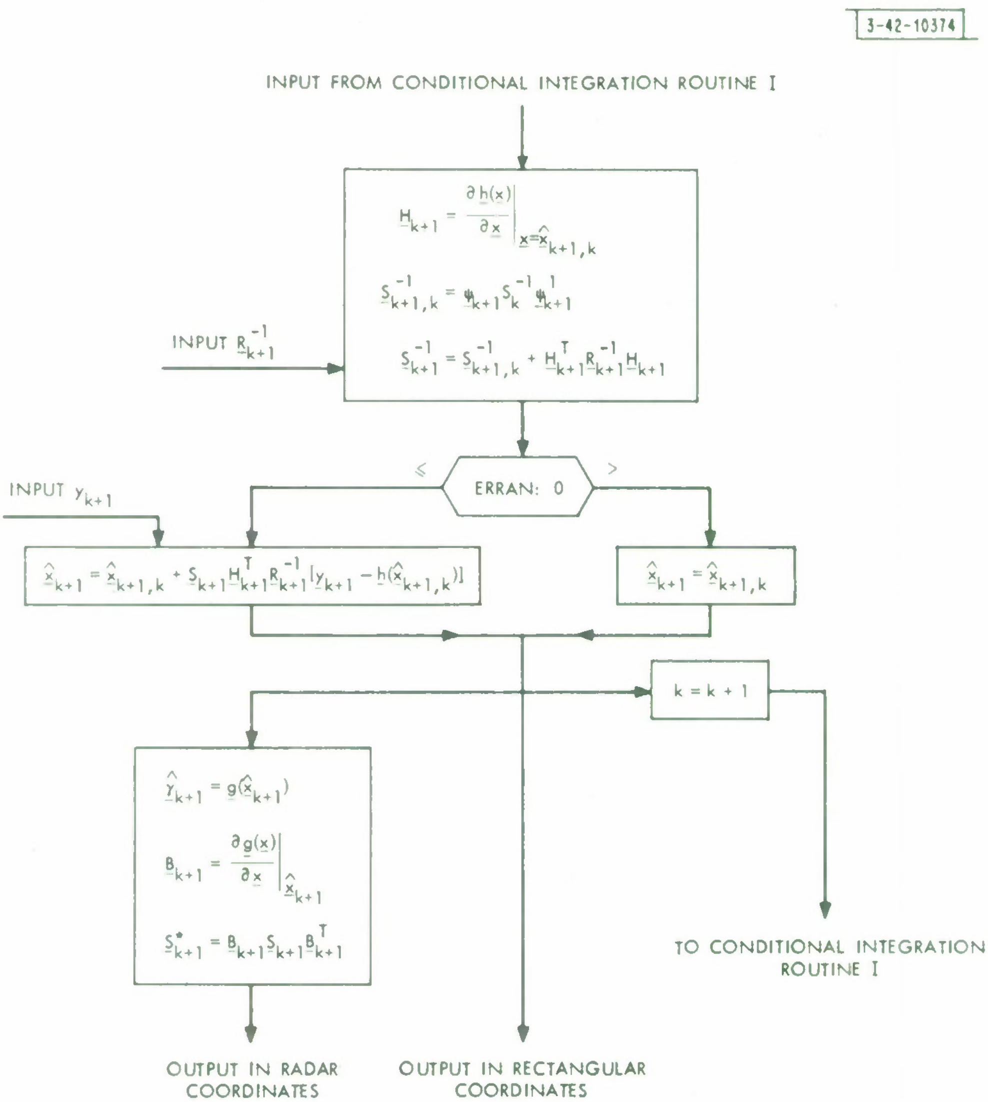

Fig. 3(a). Update routine I.

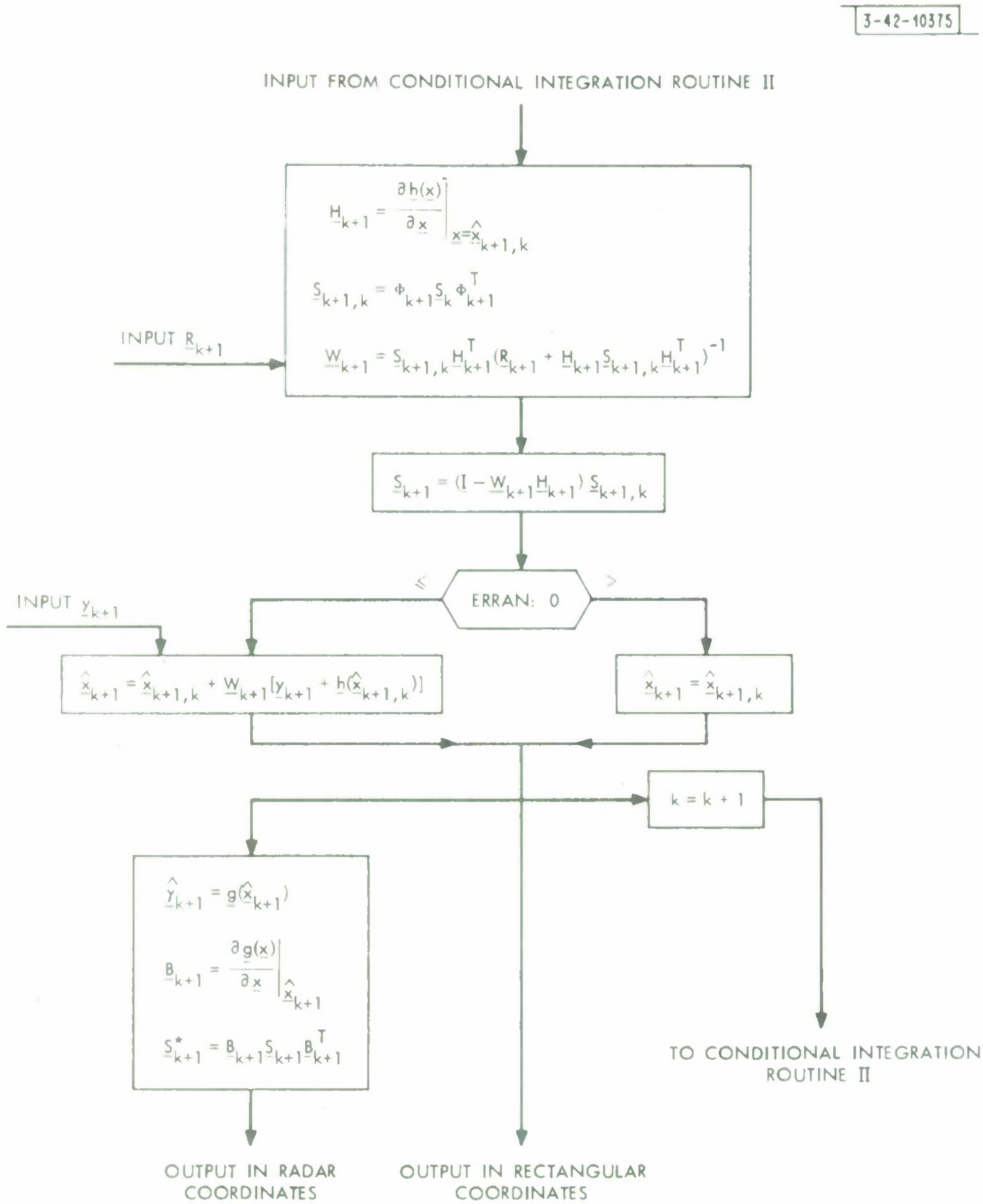

Fig. 3(b). Update routine II.

$$\underline{S}_{k+1,\,k} = E\,[(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1,\,k})\,(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1,\,k})'] = \text{covariance of the error in the estimate } \hat{\underline{x}}_{k+1,\,k}$$

$$\underline{S}_k = E\,[(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1})\,(\underline{x}_{k+1} - \hat{\underline{x}}_{k+1})'] = \text{covariance matrix for the estimate at time } t_k \quad .$$

The various blocks in Fig. 1 are described below.

A. Initial Conditions

It is assumed that initial conditions for the state estimate and the covariance matrix of that initial state estimate are known at some time t_o; it is at this time that the estimation routine starts. The first observation after (or at) t_o will produce the first estimate. The index k keeps track of the estimates $\hat{\underline{x}}_k$; the zero estimate $\hat{\underline{x}}_o$ is simply the initial estimate of the state (i.e., prior to any observation after or at t_o). The parameter ERRAN is included to allow the same program to be used for error analysis; it is set equal to one if the routine is used for error analysis and to zero otherwise.

B. Conditional Integration Routine

In the conditional integration routine, a predictor-corrector method for numerical integration is used. The specific method chosen is described in Sec. VI and will not be discussed here. However, certain modifications had to be included to permit observations at any time rather than only at an integral multiple of the step size in the numerical integration. The routine is equally well-suited for real-time estimation and post-flight analysis. If the observation times are known, they can be included as data in the program. For real-time analysis the integration will probably be one step size behind time; i.e., the state equation is integrated one step at a time only after a check is made that no observation occurred during the time interval $[t_j,\ t_j + \Delta_k]$, where t_j is the time to which the state equation has been integrated, and Δ_k is the step size for the integration in the interval $[t_k, t_{k+1}]$. If an observation is made at $t_{k+1} \epsilon [t_j,\ t_j + \Delta_k]$, the integration is carried forward in time with a step size $t_{k+1} - t_j$. Because the predictor-corrector method chosen requires a constant step size, a slightly modified predictor has to be used. In particular, this is the same predictor as the one needed to start the numerical integration algorithm. The index j keeps track of the number of integration steps, and the index ℓ is used to exit from the loop when an observation is received. The reason for having two slightly different routines [Figs. 2(a) and (b)] is given below.

C. Update Routine

The function of this routine is to update the state estimate and the covariance matrix when an observation is received. Both the updated state estimate and the covariance matrix are obtained for the rectangular coordinate system. Since it may also be of interest to have these quantities in radar coordinates, such a conversion has also been included in the routine.

The updating of the state vector is conditioned upon the parameter ERRAN. If ERRAN = 0, the routine is used for estimation, and hence the state estimate will be updated using the current observation. When an error analysis is performed, i.e., ERRAN = 1, the evolution of $\underline{S}_k$ along a given trajectory and for a given set (R_k) is found. This is accomplished by letting $\underline{x}_{k+1} = \underline{x}_{k+1,\,k}$,

as shown in Figs. 3(a) and (b). These figures represent alternate ways of solving the estimation equations as given by Mowery[4] and Cox.[5] In Fig. 3(a), $\psi = (\Phi^{-1})^T$ is computed directly, whereas in Fig. 3(b), Φ is computed first.

Section III and Refs. 4 and 5 give derivations for the state estimates and the appropriate covariance matrices. Figures 2(a) and (b) show the conditional integration routines that are used with the updating routines of Figs. 3(a) and (b), respectively. The functional form of all matrices and vectors used in the actual algorithm is given in Sec. V.

V. STATE EQUATIONS AND MATRICES DEFINED FOR THE BALLISTIC MISSILE RE-ENTRY PROBLEM

A. Dynamical Equation $\underline{f}(\underline{x})$

A spherical earth is assumed with the radar at zero altitude. The x-y-z coordinate system is centered at the radar with the x-y plane tangent to the earth (x east, y north) and z vertical to the x-y plane.

$$\frac{dx}{dt} = f_1(\underline{x}) = \dot{x}$$

$$\frac{dy}{dt} = f_2(\underline{x}) = \dot{y}$$

$$\frac{dz}{dt} = f_3(\underline{x}) = \dot{z}$$

$$\frac{d\dot{x}}{dt} = f_4(\underline{x}) = (w^2 - g_c)\,x - \frac{\rho}{2\beta}\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}\,\dot{x} + 2w\dot{y}\sin\varphi - 2w\dot{z}\cos\varphi$$

$$\frac{d\dot{y}}{dt} = f_5(\underline{x}) = (w^2\sin^2\varphi - g_c)\,y - \frac{\rho}{2\beta}\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}\,\dot{y} - 2w\dot{x}\sin\varphi$$

$$- w^2 z\sin\varphi\cos\varphi - w^2 r_o\sin\varphi\cos\varphi$$

$$\frac{d\dot{z}}{dt} = f_6(\underline{x}) = (w^2\cos^2\varphi - g_c)\,z - \frac{\rho}{2\beta}\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}\,\dot{z} - w^2 y\sin\varphi\cos\varphi$$

$$+ 2w\dot{x}\cos\varphi + r_o(w^2\cos^2\varphi - g_c)$$

$\frac{d(1/\beta)}{dt} = f_7(\underline{x}) =$ depends on the specific assumption on the functional dependence of β (= 0 for β constant)

$$g_c = \frac{G_m}{(x^2 + y^2 + z^2)^{3/2}}$$

φ = latitude of radar

r_o = radius of earth

ρ = weight density of air

β = weight-to-drag ratio

w = earth's angular rotation rate.

B. $A = \partial \underline{f}(\underline{x})/\partial \underline{x}$ Matrix

$a_{11} = a_{12} = a_{13} = 0 \quad , \qquad a_{14} = 1$

$a_{15} = a_{16} = 0 \quad , \qquad a_{21} = a_{22} = a_{23} = a_{24} = 0$

$a_{25} = 1 \quad , \qquad a_{26} = 0$

$a_{31} = a_{32} = a_{33} = a_{34} = a_{35} = 0 \quad , \qquad a_{36} = 1$

$$a_{41} = (w^2 - g_c) - x \frac{\partial g_c}{\partial x} - \dot{x}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial x} \left(\frac{\rho}{2\beta}\right)$$

$$a_{42} = -x \frac{\partial g_c}{\partial y} - \dot{x}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial y} \left(\frac{\rho}{2\beta}\right)$$

$$a_{43} = -x \frac{\partial g_c}{\partial z} - \dot{x}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial z} \left(\frac{\rho}{2\beta}\right)$$

$$a_{44} = -\frac{\rho}{2\beta} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} - \frac{\rho}{2\beta} \frac{\dot{x}^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}$$

$$a_{45} = -\frac{\rho}{2\beta} \frac{\dot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}} + 2w \sin\varphi$$

$$a_{46} = -\frac{\rho}{2\beta} \frac{\dot{x}\dot{z}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}} - 2w \cos\varphi$$

$$a_{51} = -y \frac{\partial g_c}{\partial x} - \dot{y}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial x} \left(\frac{\rho}{2\beta}\right)$$

$$a_{52} = (w^2 \sin^2\varphi - g_c) - y \frac{\partial g_c}{\partial y} - \dot{y}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial y} \left(\frac{\rho}{2\beta}\right)$$

$$a_{53} = -y \frac{\partial g_c}{\partial z} - \dot{y}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial z} \left(\frac{\rho}{2\beta}\right) - w^2 \sin\varphi \cos\varphi$$

$$a_{54} = -\frac{\rho}{2\beta} \frac{\dot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}} - 2w \sin\varphi$$

$$a_{55} = -\frac{\rho}{2\beta} \left[\frac{\dot{y}^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}} + (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}\right]$$

$$a_{56} = -\frac{\rho}{2\beta} \frac{\dot{y}\dot{z}}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}$$

$$a_{61} = -(r_o + z) \frac{\partial g_c}{\partial x} - \dot{z}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial x} \left(\frac{\rho}{2\beta}\right)$$

$$a_{62} = -(r_o + z) \frac{\partial g_c}{\partial y} - \dot{z}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial y} \left(\frac{\rho}{2\beta}\right) - w^2 \sin\varphi \cos\varphi$$

$$a_{63} = (w^2 \cos^2\varphi - g_c) - (r_o + z) \frac{\partial g_c}{\partial z} - \dot{z}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \frac{\partial}{\partial z} \left(\frac{\rho}{2\beta}\right)$$

$$a_{64} = 2w\cos\varphi - \frac{\rho}{2\beta}\frac{\dot{x}\dot{z}}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}}$$

$$a_{65} = -\frac{\rho}{2\beta}\frac{\dot{y}\dot{z}}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}}$$

$$a_{66} = -\frac{\rho}{2\beta}\left[\frac{\dot{z}^2}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}} + (\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\right] .$$

The dimension n of A as well as

$$a_{ij} \quad , \quad i = 1, 2, \ldots n; \; j > 6$$

$$a_{ij} \quad , \quad i < 6; \; j = 1, 2, \ldots n$$

will depend on the specific functional form assumed for $\alpha = 1/\beta$. The additional elements of A for α = constant are ($n = 7$, with the seventh state variable $x_7 = \alpha$):

$$a_{17} = 0 \quad , \quad a_{27} = 0 \quad , \quad a_{37} = 0$$

$$a_{47} = \frac{\rho}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\,\dot{x}$$

$$a_{57} = \frac{\rho}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\,\dot{y}$$

$$a_{67} = \frac{\rho}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{1/2}\,\dot{z}$$

$$a_{71} = a_{72} = a_{73} = a_{74} = a_{75} = a_{76} = a_{77} = 0 .$$

C. Observation Relation $\underline{h}(\underline{x})$

$$h_1(\underline{x}) = r = \text{range} = (x^2+y^2+z^2)^{1/2}$$

$$h_2(\underline{x}) = \alpha = \text{azimuth} = \tan^{-1}\frac{x}{y}$$

$$h_3(\underline{x}) = \epsilon = \text{elevation} = \tan^{-1}\frac{z}{(x^2+y^2)^{1/2}}$$

$$h_4(\underline{x}) = \dot{r} = \frac{x\dot{x}+y\dot{y}+z\dot{z}}{(x^2+y^2+z^2)^{1/2}} .$$

D. $\underline{H}$ Matrix

$$h_{11} = \frac{\partial r}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}$$

$$h_{12} = \frac{\partial r}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{1/2}} = \frac{y}{r}$$

$$h_{13} = \frac{\partial r}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}} = \frac{z}{r}$$

$$h_{14} = h_{15} = h_{16} = 0$$

$$h_{21} = \frac{\partial \alpha}{\partial x} = \frac{y}{x^2 + y^2}$$

$$h_{22} = \frac{\partial \alpha}{\partial y} = \frac{-x}{x^2 + y^2}$$

$$h_{23} = h_{24} = h_{25} = h_{26} = 0$$

$$h_{31} = \frac{\partial \epsilon}{\partial x} = \frac{-xz}{(x^2 + y^2 + z^2)(x^2 + y^2)^{1/2}} = \frac{-xz}{r^2(x^2 + y^2)^{1/2}}$$

$$h_{32} = \frac{\partial \epsilon}{\partial y} = \frac{-yz}{(x^2 + y^2 + z^2)(x^2 + y^2)^{1/2}} = \frac{-yz}{r^2(x^2 + y^2)^{1/2}}$$

$$h_{33} = \frac{\partial \epsilon}{\partial z} = \frac{(x^2 + y^2)^{1/2}}{x^2 + y^2 + z^2} = \frac{(x^2 + y^2)^{1/2}}{r^2}$$

$$h_{34} = h_{35} = h_{36} = 0$$

$$h_{41} = \frac{\partial \dot{r}}{\partial x} = \frac{\dot{x}(x^2 + y^2 + z^2) - x(x\dot{x} + y\dot{y} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\dot{x}r - \dot{r}x}{r^2}$$

$$h_{42} = \frac{\partial \dot{r}}{\partial y} = \frac{\dot{y}(x^2 + y^2 + z^2) - y(x\dot{x} + y\dot{y} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\dot{y}r - \dot{r}y}{r^2}$$

$$h_{43} = \frac{\partial \dot{r}}{\partial z} = \frac{\dot{z}(x^2 + y^2 + z^2) - z(x\dot{x} + y\dot{y} + z\dot{z})}{(x^2 + y^2 + z^2)^{3/2}} = \frac{\dot{z}r - \dot{r}z}{r^2}$$

$$h_{44} = \frac{\partial \dot{r}}{\partial \dot{x}} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r}$$

$$h_{45} = \frac{\partial \dot{r}}{\partial \dot{y}} = \frac{y}{(x^2 + y^2 + z^2)^{1/2}} = \frac{y}{r}$$

$$h_{46} = \frac{\partial \dot{r}}{\partial \dot{z}} = \frac{z}{(x^2 + y^2 + z^2)^{1/2}} = \frac{z}{r} \quad .$$

E. Relation Between Coordinate Systems $\underline{g}(\underline{x})^\dagger$

$$g_1(\underline{x}) = r = (x^2 + y^2 + z^2)^{1/2}$$

$$g_2(\underline{x}) = \alpha = \tan^{-1} \frac{x}{y}$$

$$g_3(\underline{x}) = \epsilon = \tan^{-1} \frac{z}{(x^2 + y^2)^{1/2}}$$

$$g_4(\underline{x}) = \dot{r} = \frac{x\dot{x} + y\dot{y} + z\dot{z}}{(x^2 + y^2 + z^2)^{1/2}}$$

† $\underline{g}(\underline{x})$ as defined here bears no relation to $\underline{g}(\underline{x})$ in (3-1).

$$g_5(\underline{x}) = \dot{\alpha} = \frac{y\dot{z} - x\dot{y}}{(x^2 + y^2)}$$

$$g_6(\underline{x}) = \dot{\epsilon} = \frac{(x^2 + y^2)\,\dot{z} - z(x\dot{x} + y\dot{y})}{(x^2 + y^2 + z^2)\,(x^2 + y^2)^{1/2}} \quad .$$

F. $\underline{B} = \partial \underline{g}(\underline{x})/\partial \underline{x}$ Matrix

$$b_{11} = \frac{x}{r} \;, \qquad b_{12} = \frac{y}{r} \;, \qquad b_{13} = \frac{z}{r}$$

$$b_{14} = 0 \;, \qquad b_{15} = 0 \;, \qquad b_{16} = 0$$

$$b_{21} = \frac{y}{x^2 + y^2} \;, \qquad b_{22} = \frac{-x}{x^2 + y^2} \;, \qquad b_{23} = 0$$

$$b_{24} = 0 \;, \qquad b_{25} = 0 \qquad b_{26} = 0$$

$$b_{31} = \frac{-xz}{r^2(x^2 + y^2)^{1/2}} \;, \qquad b_{32} = \frac{-yz}{r^2(x^2 + y^2)^{1/2}} \;, \qquad b_{33} = \frac{x^2 + y^2}{r^2}$$

$$b_{34} = 0 \;, \qquad b_{35} = 0 \;, \qquad b_{36} = 0$$

$$b_{41} = \frac{r\dot{x} - x\dot{r}}{r^2} \;, \qquad b_{42} = \frac{r\dot{y} - y\dot{r}}{r^2} \;, \qquad b_{43} = \frac{r\dot{z} - z\dot{r}}{r^2}$$

$$b_{44} = \frac{x}{r} \;, \qquad b_{45} = \frac{y}{r} \;, \qquad b_{46} = \frac{z}{r}$$

$$b_{51} = \frac{2x\dot{\alpha} + \dot{y}}{x^2 + y^2} \;, \qquad b_{52} = \frac{\dot{x} - 2\dot{\alpha}y}{x^2 + y^2} \;, \qquad b_{53} = 0$$

$$b_{54} = \frac{y}{x^2 + y^2} \;, \qquad b_{55} = \frac{-x}{x^2 + y^2} \;, \qquad b_{56} = 0$$

$$b_{61} = \frac{-x\dot{\epsilon}}{x^2 + y^2} - \frac{2\dot{r}}{r}\, b_{31} - \frac{z\dot{x}}{r^2(x^2 + y^2)^{1/2}}$$

$$b_{62} = \frac{y\dot{\epsilon}}{x^2 + y^2} - \frac{2\dot{r}}{r}\, b_{32} - \frac{z\dot{y}}{r^2(x^2 + y^2)^{1/2}}$$

$$b_{63} = -\frac{z\dot{\epsilon}}{r^2} - \frac{\dot{r}}{r}\, b_{33} \;, \qquad b_{64} = b_{31} = \frac{-xz}{r^2(x^2 + y^2)^{1/2}}$$

$$b_{65} = b_{32} = \frac{-yz}{r^2(x^2 + y^2)^{1/2}} \;, \qquad b_{66} = b_{33} = \frac{x^2 + y^2}{r^2} \quad .$$

VI. NUMERICAL INTEGRATION OF THE EQUATIONS OF MOTION

In Sec. II we considered a system described by the nonlinear state differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \tag{6-1}$$

where

$$\underline{x}(t_o) = \underline{x}_o$$

and $\underline{x}$ is an n-dimensional state vector. Discussion was limited to the general concept of linearizing the nonlinear equations about a nominal trajectory. In this section these concepts will be extended to obtain a specific algorithm for solving the system of equations presented in Sec. V. This algorithm is then used to compute the $\hat{\underline{x}}_{k,\,k-1}$ or conditional estimate term of (3-23).

In addition, there is a requirement to update the covariance matrix (2-9) or (3-25). The integration algorithm given here satisfies this need. Part D presents details concerning computation of the incrementally linearized transition matrix used in updating covariances.

Let the solution of (6-1) with the initial condition $\underline{x}_o$ be denoted by $\underline{x}^o(t)$. There are a number of schemes available which use numerical integration to obtain the solution $\underline{x}^o(t)$. Since in the present case it will also be necessary to find a solution to the linearized incremental differential equation

$$\frac{d}{dt}\,[\delta \underline{x}(t)] = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}^o(t)} \delta \underline{x}(t) \tag{6-2}$$

only methods which make use of higher than first derivatives (in particular, second derivatives) of $\underline{x}(t)$ at various times will be considered.

A. Taylor Series Predictor

The simplest way of using the knowledge of $\ddot{\underline{x}}$ is by means of a Taylor series expansion. Assume that the value of $\underline{x}(t)$ is known at time t_k. For simplicity, it will be assumed that $\underline{x}(t)$ will be found at equally spaced points along the time axis. Let

$$\Delta = t_k - t_{k-1} = \text{constant for all } k$$

$$\underline{x}_k = \underline{x}(t_k)$$

$$\dot{\underline{x}}_k = \left.\frac{d\underline{x}}{dt}\right|_{t=t_k} = \underline{f}(\underline{x}_k)$$

then

$$\underline{x}_{k+1} = \underline{x}_k + \dot{\underline{x}}_k \Delta + \ddot{\underline{x}}_k \frac{\Delta^2}{2} + \text{higher order terms}$$

since

$$\ddot{\underline{x}}_k = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}_k} \dot{\underline{x}}_k = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}_k} \underline{f}(\underline{x}_k)$$

$$\underline{x}_{k+1} = \underline{x}_k + \underline{f}(\underline{x}_k)\,\Delta + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}_k} \underline{f}(\underline{x}_k)\,\frac{\Delta^2}{2} + \text{higher order terms} \quad .$$

Hence, $\underline{x}_{k+1}$ can be approximated by

$$\underline{x}_{k+1} = \underline{x}_k + \underline{f}(\underline{x}_k)\,\Delta + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}_k} \underline{f}(\underline{x}_k)\,\frac{\Delta^2}{2} \tag{6-3}$$

with the i^{th} component of error $E_i = (\Delta^3/3)\, x_i^{(3)}(\sigma_i)\; t_k \leqslant \sigma_i \leqslant t_{k+1}$.

B. Predictors Based on More Than One Point

If only the values of $\underline{x}_k$ and its derivatives are to be used, then the Taylor series expansion is the best polynomial approximation. However, by using appropriate previous values of $\underline{x}$ and its derivatives, an improvement over (6-3) is possible. A number of such predictors exist which can be generated in several ways. Four predictors are given below, together with the derivation for the predictor used in the algorithm described in part D.

The following predictors involve only $\underline{x}_{k-1}, \dot{\underline{x}}_{k-1}, \underline{x}_k, \dot{\underline{x}}_k, \ddot{\underline{x}}_k, \ddot{\underline{x}}_{k-1}$:

$$\underline{x}_{k+1} = \underline{x}_k + \frac{\Delta}{2}(3\dot{\underline{x}}_{k-1} - \dot{\underline{x}}_k) + \frac{\Delta^2}{12}(17\ddot{\underline{x}}_k + 7\ddot{\underline{x}}_{k-1}) \tag{6-4}$$

$$\underline{x}_{k+1} = \frac{1}{2}(\underline{x}_k + \underline{x}_{k-1}) + \frac{\Delta}{4}(7\dot{\underline{x}}_{k-1} - \dot{\underline{x}}_k) + \frac{\Delta^2}{24}(33\ddot{\underline{x}}_k + 15\ddot{\underline{x}}_{k-1}) \tag{6-5}$$

$$\underline{x}_{k+1} = \underline{x}_{k-1} + 2\Delta\dot{\underline{x}}_{k-1} + \frac{\Delta^2}{3}(4\ddot{\underline{x}}_k + 2\ddot{\underline{x}}_{k-1}) \quad . \tag{6-6}$$

A predictor using $\underline{x}_{k-2}, \underline{x}_{k-1}, \underline{x}_k, \ddot{\underline{x}}_{k-1}, \ddot{\underline{x}}_k$ is

$$\underline{x}_{k+1} = \underline{x}_{k-2} + 3(\underline{x}_k - \underline{x}_{k-1}) + \Delta^2(\ddot{\underline{x}}_k - \ddot{\underline{x}}_{k-1}) \quad . \tag{6-7}$$

To derive (6-6) we use the Taylor series expansion for $\underline{x}_{k+1}, \underline{x}_{k-1}$ and its first and second derivatives. Equations (6-4), (6-5), and (6-7) can be derived similarly, and in fact, by using the same method, an infinite variety of predictors can be generated. The various series are [assuming $\underline{x}^{(5)}(s)$ continuous]

$$\underline{x}_{k+1} = \underline{x}_k + \Delta\dot{\underline{x}}_k + \frac{\Delta^2}{2}\ddot{\underline{x}}_k + \frac{\Delta^3}{3!}\underline{x}_k^{(3)} + \frac{\Delta^4}{4!}\underline{x}_k^{(4)} + \underline{R}_4 \tag{6-8}$$

where the i^{th} component of $\underline{R}_4$ is given by

$$[\underline{R}_4]_i = \frac{1}{4!}\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,(t_{k+1} - s)^4\, ds \tag{6-9}$$

$$\underline{x}_{k-1} = \underline{x}_k - \Delta\dot{\underline{x}}_k + \frac{\Delta^2}{2}\ddot{\underline{x}}_k - \frac{\Delta^3}{3!}\underline{x}_k^{(3)} + \frac{\Delta^4}{4!}\underline{x}_k^{(4)} + \underline{R}'_4 \tag{6-10}$$

$$[\underline{R}'_4]_i = \frac{1}{4!}\int_{t_k}^{t_{k-1}} x_i^{(5)}(s)\,(t_{k-1} - s)^4\, ds \tag{6-11}$$

and

$$\Delta\dot{\underline{x}}_{k-1} = \Delta\dot{\underline{x}}_k - \Delta^2\ddot{\underline{x}}_k + \frac{\Delta^3}{2!}\underline{x}_k^{(3)} - \frac{\Delta^4}{3!}\underline{x}_k^{(4)} + \underline{R}''_4 \tag{6-12}$$

$$[\underline{R}''_4]_i = \frac{\Delta}{3!}\int_{t_k}^{t_{k-1}} x_i^{(5)}(s)\,(t_{k-1} - s)^3\, ds \tag{6-13}$$

$$\Delta^2\ddot{\underline{x}}_{k-1} = \Delta^2\ddot{\underline{x}}_k - \Delta^3\underline{x}_k^{(3)} + \frac{\Delta^4}{2!}\underline{x}_k^{(4)} + \underline{R}'''_4 \tag{6-14}$$

$$[\underline{R}_4''']_i = \frac{\Delta^2}{2!} \int_{t_k}^{t_{k-1}} x_i^{(5)}(s)\,(t_{k-1} - s)^2\, ds \quad . \tag{6-15}$$

Multiplying (6-10) by -1, (6-12) by -2, (6-14) by $-2/3$, and adding them to (6-8) yields

$$\underline{x}_{k+1} - \underline{x}_{k-1} - 2\Delta\dot{\underline{x}}_{k-1} - \frac{2}{3}\Delta^2\ddot{\underline{x}}_{k-1} = \frac{4}{3}\Delta^2\ddot{\underline{x}}_k + \underline{R}_4 - \underline{R}_4' - 2\underline{R}_4'' - \frac{2}{3}\underline{R}_4'''$$

or

$$\underline{x}_{k+1} = \underline{x}_{k-1} + 2\Delta\dot{\underline{x}}_{k-1} + \frac{2}{3}\Delta^2(\ddot{\underline{x}}_{k-1} + 2\ddot{\underline{x}}_k) + \underline{R}_4^* \tag{6-16}$$

where

$$\underline{R}_4^* = \underline{R}_4 - \underline{R}_4' - 2\underline{R}_4'' - \frac{2}{3}\underline{R}_4'''$$

$$[\underline{R}_4^*]_i = \frac{1}{4!}\left\{\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,(t_{k+1} - s)^4\, ds - \int_{t_k}^{t_{k-1}} x_i^{(5)}(s)\,[(t_{k-1} - s)^4 \right.$$

$$\left. + 8\Delta(t_{k-1} - s)^3 + 8\Delta^2(t_{k-1} - s)^2]\, ds\right\} \quad . \tag{6-17}$$

Consider now the term

$$[(t_{k-1} - s)^4 + 8\Delta(t_{k-1} - s)^3 + 8\Delta^2(t_{k-1} - s)^2] \quad . \tag{6-18}$$

In (6-18) it is clear that since $(t_{k-1} - s) \leqslant \Delta$

$$8\Delta(t_{k-1} - s)^3 + 8\Delta^2(t_{k-1} - s)^2 \geqslant 0 \quad , \quad t_{k-1} \leqslant s \leqslant t_k \quad .$$

Hence, (6-18) is non-negative in the interval of integration. By using the mean value theorem of integral calculus, (6-17) can be written in the form

$$[\underline{R}_4^*]_i = \frac{1}{4!} x_i^{(5)}(\sigma_i) \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^4\, ds + \frac{1}{4!} x_i^{(5)}(\varphi_i) \int_{t_{k-1}}^{t_k}$$

$$\times [(t_{k-1} - s)^4 + 8\Delta(t_{k-1} - s)^3 + 8\Delta^2(t_{k-1} - s)^2]\, ds$$

$$= \frac{\Delta^5}{5!} x_i^{(5)}(\sigma_i) + \frac{\Delta^5}{5!} x_i^{(5)}(\varphi_i)\left(1 - \frac{8 \times 5}{4} + \frac{8 \times 5}{3}\right)$$

or

$$[\underline{R}_4^*]_i = \frac{\Delta^5}{5!}\left[x_i^{(5)}(\sigma_i) + \frac{13}{3} x_i^{(5)}(\varphi_i)\right] \tag{6-19}$$

where $t_k \leqslant \sigma_i \leqslant t_{k+1}$; $t_{k-1} \leqslant \varphi_i \leqslant t_k$ for $i = 1, 2, \ldots n$. Assume now that

$$x_i^{(5)}(\sigma_i) = x_i^{(5)}(\varphi_i) + h_i$$

then

$$[\underline{R}_4^*]_i = \frac{\Delta^5}{5! \times 3} [3x_i^{(5)}(\varphi_i) + 3h + 13x_i^{(5)}(\varphi_i)]$$

$$= \frac{\Delta^5}{5! \times 3} [16x_i^{(5)}(\varphi_i) + 3h_i]$$

$$= \frac{16\Delta^5}{360} [x_i^{(5)}(\varphi_i) + \frac{3}{16} h_i]$$

but $x_i^{(5)}(\varphi_i) + 3/16\ h_i$ will be a value between $x_i^{(5)}(\varphi_i)$ and $x_i^{(5)}(\sigma_i)$. Hence, because of continuity, there exists a ψ such that

$$[\underline{R}_4^*]_i = \frac{2}{45} x_i^{(5)}(\psi_i)\, \Delta^5$$

$$t_{k-1} \leqslant \psi_i \leqslant t_{k+1} \qquad \text{for } i = 1, \ldots n \quad . \tag{6-20}$$

Hence, if the predictor (6-6)

$$\underline{x}_{k+1} = \underline{x}_{k-1} + 2\Delta\dot{\underline{x}}_{k-1} + \frac{2}{3}\Delta^2(\ddot{\underline{x}}_{k-1} + 2\ddot{\underline{x}}_k)$$

is used, the remainder (or error) term is given by (6-20).

C. Correctors

By means of iterations it is possible to improve the accuracy of numerical integration. Formulas used for the iterations are called correctors, since they all require a starting value which is then "corrected" in subsequent iterations. To obtain such formulas a procedure identical to that in the previous section can be used. In particular

$$\underline{x}_{k+1} = \underline{x}_k + \Delta\dot{\underline{x}}_k + \frac{\Delta^2}{2}\ddot{\underline{x}}_k + \frac{\Delta^3}{3!}\underline{x}_k^{(3)} + \frac{\Delta^4}{4!}\underline{x}_k^{(4)} + \underline{R}_5 \tag{6-21}$$

$$[\underline{R}_5]_i = \frac{1}{4!}\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,(t_{k+1} - s)^4\, ds \tag{6-22}$$

and

$$\Delta\dot{\underline{x}}_{k+1} = \Delta\dot{\underline{x}}_k + \Delta^2\ddot{\underline{x}}_k + \frac{\Delta^3}{2}\underline{x}_k^{(3)} + \frac{\Delta^4}{3!}\underline{x}_k^{(4)} + \underline{R}_5' \tag{6-23}$$

$$[\underline{R}_5']_i = \frac{1}{3!}\Delta\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,(t_{k+1} - s)^3\, ds \tag{6-24}$$

$$\Delta^2\ddot{\underline{x}}_{k+1} = \Delta^2\ddot{\underline{x}}_k + \Delta^3\underline{x}_k^{(3)} + \frac{\Delta^4}{2}\underline{x}_k^{(4)} + \underline{R}_5'' \tag{6-25}$$

$$[\underline{R}_5'']_i = \frac{\Delta^2}{2!}\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,(t_{k+1} - s)^2\, ds \quad . \tag{6-26}$$

Multiplying (6-23) by $-1/2$, (6-25) by $1/12$, and adding them to (6-21) yields

$$\underline{x}_{k+1} = \underline{x}_k + \frac{\Delta}{2}(\dot{\underline{x}}_k + \dot{\underline{x}}_{k+1}) + \frac{1}{12}\Delta^2(\ddot{\underline{x}}_k - \ddot{\underline{x}}_{k+1}) + \underline{R}_5^* \tag{6-27}$$

$$\underline{R}_5^* = \underline{R}_5 - \frac{1}{2}\underline{R}_5' + \frac{1}{12}\underline{R}_5''$$

$$[\underline{R}_5^*]_i = \frac{1}{4!}\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,[(t_{k+1}-s)^4 - 2\Delta(t_{k+1}-s)^3 + \Delta^2(t_k-s)^2]\,ds$$

$$= \frac{1}{4!}\int_{t_k}^{t_{k+1}} x_i^{(5)}(s)\,[(t_{k+1}-s)^2 - \Delta(t_{k+1}-s)]^2\,ds \quad . \tag{6-28}$$

Again applying the mean value theorem of integral calculus to (6-28) yields

$$[\underline{R}_5^*]_i = \frac{1}{4!}\,x_i^{(5)}(\beta_i)\left(\frac{\Delta^5}{5} - \frac{\Delta^5}{2} + \frac{\Delta^5}{3}\right)$$

or

$$[\underline{R}_5^*]_i = \frac{\Delta^5}{720}\,x_i^{(5)}(\beta_i) \tag{6-29}$$

where $t_k \leqslant \beta_i \leqslant t_{k+1}$ for $i = 1, \ldots n$.

Hence, if

$$\underline{x}_{k+1} = \underline{x}_k + \frac{\Delta}{2}(\dot{\underline{x}}_k + \dot{\underline{x}}_{k+1}) + \frac{1}{12}\Delta^2(\ddot{\underline{x}}_k - \ddot{\underline{x}}_{k+1}) \tag{6-30}$$

is used as the corrector, the remaining (or error) terms are given by (6-29).

D. Algorithm for the Numerical Solution of a Differential Equation

It is now possible to give a complete algorithm for the solution of the differential equation

$$\dot{\underline{x}} = \underline{f}(\underline{x})$$

$$\underline{x}(0) = \underline{x}_o \quad . \tag{6-31}$$

The predictor-corrector method is used, with the corrector being of the form of (6-30). For the predictor, (6-6) was selected. Recall from Sec. II that it is also necessary to evaluate the transition matrix for the linear time-varying equation

$$\frac{d}{dt} = (\partial\underline{x})\left.\frac{\partial\underline{f}}{\partial\underline{x}}\right|_{\underline{x}^o(t)}\delta\underline{x} \tag{6-32}$$

in order to update the covariance matrix. This is easily incorporated into the present algorithm. The flow chart (Fig. 4) shows the sequence of operations. Since (6-6) requires $\underline{x}_{k-1}$, $\underline{x}_1$ has to be evaluated by other means. As indicated in the flow chart, this is done by using a Taylor series expansion about $\underline{x}_o$.

In the flow chart symbolism, Φ denotes the transition matrix of (6-32). For evaluating Φ, the assumption is made that during the interval of time of length Δ, $\partial\underline{f}/\partial\underline{x}$ will be essentially constant. Hence if the notation

$$\left.\frac{\partial\underline{f}}{\partial\underline{x}}\right|_{x_k} \equiv \underline{A}_k \tag{6-33}$$

is used, the transition matrix at the k^{th} point

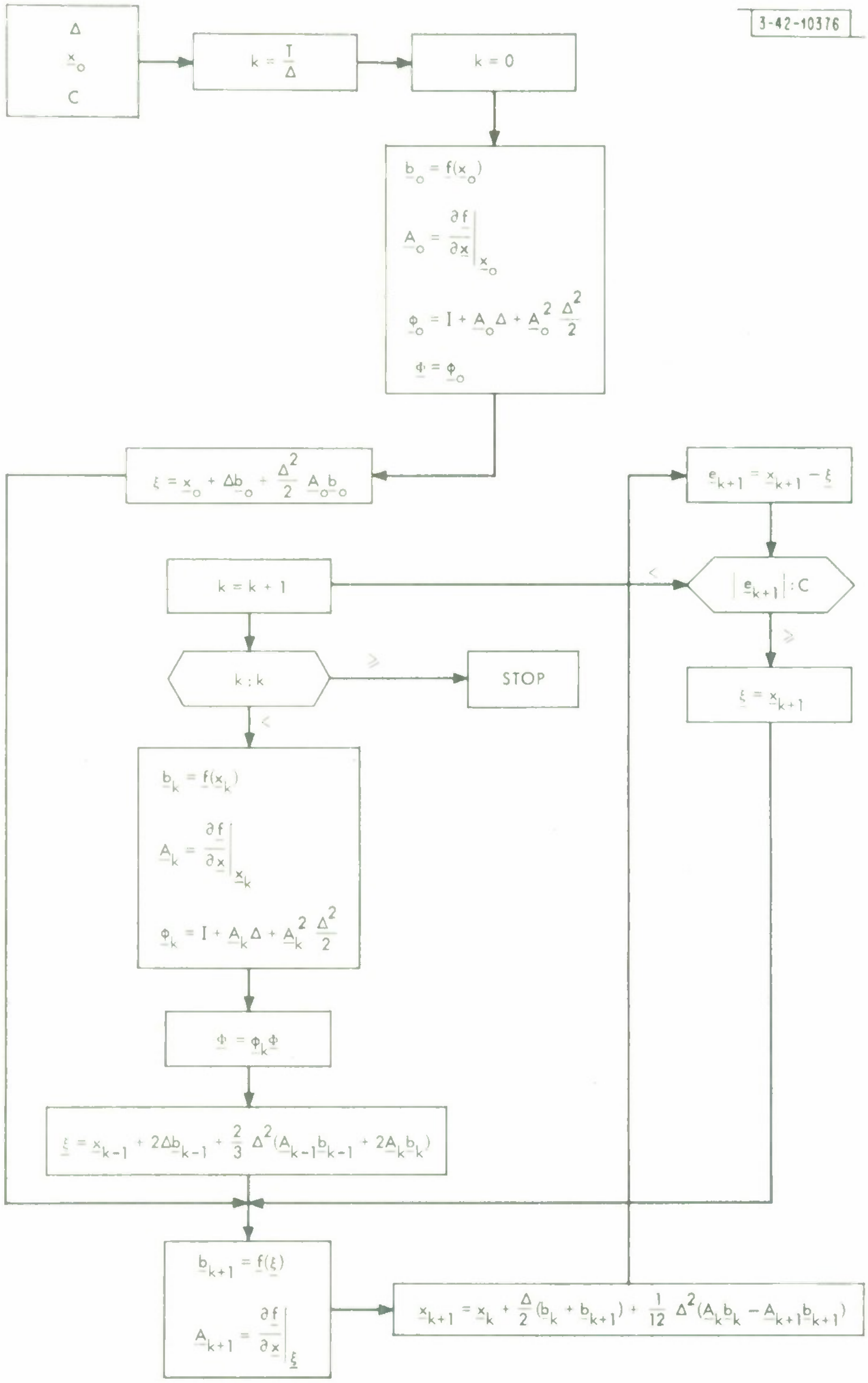

Fig. 4. Flow chart of algorithm.

$$\Phi = \phi(t_k, t_o) \tag{6-34}$$

can be written as

$$\Phi = e^{\underline{A}_k \Delta} e^{\underline{A}_{k-1} \Delta} \ldots e^{\underline{A}_o \Delta} \quad . \tag{6-35}$$

This follows from the fact that with the assumption of constant $\partial \underline{f}/\partial \underline{x}$ during the interval Δ, (6-32) represents a piecewise constant system. It is also easy to see that

$$\phi(t_{k+1}, t_o) = \phi(t_{k+1}, t_k)\, \phi(t_k, t_o) = e^{\underline{A}_{k+1}\Delta}\, \phi(t_k, t_o) \quad . \tag{6-36}$$

Equation (6-36) updates Φ at each new t_k. In order to evaluate $\exp[\underline{A}_{k+1}\Delta]$, a power series expansion of the form $\exp[\underline{A}_k \Delta] = 1 + \underline{A}_k \Delta + \underline{A}_k^2(\Delta^2/2)$ + (higher order terms) was used, with the higher order terms neglected. For Δ sufficiently small (which it has to be for the assumption $\partial \underline{f}/\partial \underline{x}$ constant), this approximation will be valid. However, for increased accuracy we may

(1) Drop the assumption $(\partial \underline{f}/\partial \underline{x})\Big|_{\underline{x}^o(t)}$ constant during the interval Δ. As an approximation it could be assumed for example that in the interval $[t_k, t_{k+1}]$, (6-32) takes the form

$$\frac{d}{dt}[\delta \underline{x}(t)] = A_k + \frac{(\underline{A}_{k+1} - \underline{A}_k)}{\Delta}(t - t_k)\, \delta \underline{x}(t) \quad . \tag{6-37}$$

(2) Evaluate the transition matrix $\phi(t_{k+1}, t_k) = \phi_k$ to powers in t higher than the second.

It will be seen that in the flow diagram the correction iteration loop is made conditional upon the size of the difference between successive approximations. This need not be incorporated, although it is worthwhile to obtain and record the difference between the predicted value and the value found by iteration. This will become clear in the next section.

E. Estimate of the Truncation Error

In the algorithm described previously, (6-6) was used as the predictor, and (6-30) was used to obtain the final value of $\underline{x}_{k+1}$. If we denote the difference between the corrected and the predicted value of $\underline{x}_{k+1}$ by $\underline{C}_o$, then $\underline{C}_o$ represents the difference in the truncation error of (6-30) and (6-6) or

$$\underline{C}_o = \text{predicted value} - \text{corrected value}$$

$$= \underline{R}_4^* - \underline{R}_5^*$$

$$[\underline{C}_o]_i = \frac{32}{720}\Delta^5 x_i^{(5)}(\psi_i) - \frac{\Delta^5}{720} x_i^{(5)}(\beta_i) \tag{6-38}$$

where it will be recalled that $t_{k-1} \leqslant \psi_i \leqslant t_{k+1}$ and $t_k \leqslant \beta_i \leqslant t_{k+1}$ or

$$[\underline{C}_o]_i = \frac{31}{720}\Delta^5 x_i^{(5)}(\psi_i) + \frac{\Delta^5}{720} x_i^{(6)}(\alpha_i)(\psi_i - \beta_i) \tag{6-39}$$

where $x_i^{(5)}(\psi_i) - x_i^{(5)}(\beta_i)$ was expressed as

$$x_i^{(5)}(\psi_i) - x_i^{(5)}(\beta_i) = \int_{\beta_i}^{\psi_i} x_i^{(6)}(s)\, ds = x_i^{(6)}(\alpha_i)(\psi_i - \beta_i)$$

and $\min(\psi_i, \beta_i) \leqslant \alpha_i \leqslant \max(\psi_i, \beta_i)$. Hence

$$\frac{[\underline{C}_o]_i}{31} = \frac{1}{720}\Delta^5 x_i^{(5)}(\psi_i) + \frac{\Delta^5}{31 \times 720} x_i^{(6)}(\alpha_i)(\psi_i - \beta_i)$$

and

$$\left| [\underline{R}_5^*]_i - \left[\frac{\underline{C}_o}{31}\right]_i \right| = \left| \frac{\Delta^5}{720} [x_i^{(5)}(\beta_i) - x_i^{(5)}(\psi_i) - \frac{1}{31} x_i^{(6)}(\alpha_i)(\psi_i - \beta_i)] \right|$$

$$= \frac{\Delta^5}{720} \left| -\frac{32}{31} x_i^{(6)}(\alpha_i)(\psi_i - \beta_i) \right|$$

$$\leqslant \frac{\Delta^5}{720} M \frac{32}{31} |\psi_i - \beta_i| \leqslant \frac{\Delta^6}{720} M \frac{32}{31}$$

where it is assumed that

$$|\underline{x}^{(6)}(t)| \leqslant M < \infty \quad , \quad t_{k-1} \leqslant t \leqslant t_{k+1} \quad .$$

Thus

$$\left| \underline{R}_5^* - \frac{\underline{C}_o}{31} \right| \leqslant \sum_{i=1}^{N} \left| [R_5^*]_i - \left[\frac{\underline{C}_o}{31}\right]_i \right| \leqslant \frac{2NM\Delta^6}{1395} \quad . \tag{6-40}$$

From (6-4) it follows that

$$\underline{R}_5^* \cong \frac{\underline{C}_o}{31} \tag{6-41}$$

with an error of the order Δ^6.

In general, Δ will be chosen small enough so that only one iteration is required. Consequently, a good estimate of the truncation error (with an accuracy of order $\underline{C}_o$) can be obtained by recording e_{k+1} for the first pass through the iteration loop (see Fig. 4).

F. Convergence of Iterations

Previously, it has been tacitly assumed that the iterations indicated in the algorithm will converge. To find under what circumstances this will be true, consider the i^{th} iteration. Here superscripts will denote the value of $\underline{x}_{k+1}$ obtained at the respective iteration, i.e.,

$$\underline{x}_{k+1}^{i+1} = \underline{x}_k + \frac{\Delta}{2}(\dot{\underline{x}}_k + \dot{\underline{x}}_{k+1}^{i}) + \frac{1}{12}\Delta^2(\ddot{\underline{x}}_k - \ddot{\underline{x}}_{k+1}^{i})$$

represents the value of $\underline{x}_{k+1}$ after the i^{th} iteration. Then

$$\underline{x}_{k+1}^{i+1} - \underline{x}_{k+1}^{i} = \frac{\Delta}{2}(\dot{\underline{x}}_{k+1}^{i} - \dot{\underline{x}}_{k+1}^{i-1}) + \frac{1}{12}\Delta^2(\ddot{\underline{x}}_{k+1}^{i-1} - \ddot{\underline{x}}_{k+1}^{i}) \quad .$$

Let $\underline{\delta}_j = \underline{x}^j_{k+1} - \underline{x}^{j-1}_{k+1}$, then

$$\underline{\delta}_{i+1} = \frac{\Delta}{2} \frac{d}{dt} (\underline{\delta}_i) - \frac{\Delta^2}{12} \frac{d^2}{dt^2} (\underline{\delta}_i) \quad . \tag{6-42}$$

In order to obtain $d\underline{\delta}_i/dt$ and $d^2\underline{\delta}_i/dt^2$ in terms of $\underline{\delta}_i$, the differential equation of motion (6-31) will be used.

$$\dot{\underline{x}}^i_{k+1} = \underline{f}(\underline{x}^i_{k+1}) \quad .$$

Hence

$$\dot{\underline{x}}^i_{k+1} - \dot{\underline{x}}^{i-1}_{k+1} = \underline{f}(\underline{x}^i_{k+1}) - \underline{f}(\underline{x}^{i-1}_{k+1}) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{\xi}_i} (\underline{x}^i_{k+1} - \underline{x}^{i-1}_{k+1}) \tag{6-43}$$

or

$$\frac{d}{dt} (\underline{\delta}_i) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{\xi}_i} \underline{\delta}_i$$

where $\partial \underline{f}/\partial \underline{x}$ is assumed continuous in the interval of consideration, and $\underline{\xi}_i$ is a value of $\underline{x}$ such that (6-43) is satisfied. To simplify notation, the definition

$$\underline{F}(\underline{\xi}_i) \equiv \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{\xi}_i} \tag{6-44}$$

is introduced. To obtain the variation in the second derivative, the expression for $\ddot{\underline{x}}_{k+1}$ will first be obtained. It follows from (6-31) that

$$\ddot{\underline{x}}_{k+1} = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}_{k+1}} \dot{\underline{x}}_{k+1} = \left.\frac{\partial \underline{f}}{\partial \underline{x}} \underline{f}(\underline{x})\right|_{\underline{x}=\underline{x}_{k+1}} \quad . \tag{6-45}$$

Hence

$$\ddot{\underline{x}}^i_{k+1} = \underline{g}(\underline{x}^i_{k+1}) \tag{6-46}$$

where $\underline{g}(\cdot)$ is defined by

$$\underline{g}(\underline{x}) = \frac{\partial \underline{f}(\underline{x})}{\partial \underline{x}} \underline{f}(\underline{x}) \quad .$$

Therefore

$$\ddot{\underline{x}}^i_{k+1} - \ddot{\underline{x}}^{i-1}_{k+1} = \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\underline{x}=\underline{\eta}_i} (\underline{x}^i_{k+1} - \underline{x}^{i-1}_{k+1}) \tag{6-47}$$

where it is again assumed that $\partial \underline{g}/\partial \underline{x}$ is continuous, and $\underline{\eta}_i$ is chosen to satisfy (6-47). Defining

$$\underline{G}(\underline{\eta}_i) = \left.\frac{\partial \underline{g}}{\partial \underline{x}}\right|_{\underline{x}=\underline{\eta}_i} \tag{6-48}$$

(6-47) becomes

$$\frac{d^2}{dt^2}(\underline{\delta}_i) = \underline{G}(\underline{\eta}_i)\,\underline{\delta}_i \quad .$$

Hence, (6-42) becomes

$$\underline{\delta}_{i+1} = \left[\frac{\Delta}{2}\,\underline{F}(\underline{\xi}_i) - \frac{\Delta^2}{12}\,\underline{G}(\underline{\eta}_i)\right]\underline{\delta}_i \quad . \tag{6-49}$$

Equation (6-49) represents a vector difference equation of the form

$$\underline{\delta}_{i+1} = \underline{A}_i\underline{\delta}_i \tag{6-50}$$

$$\underline{A}_i \equiv \frac{\Delta}{2}\,\underline{F}(\underline{\xi}_i) - \frac{\Delta^2}{12}\,\underline{G}(\underline{\eta}_i) \quad .$$

If all the $\underline{\delta}_i$'s are small enough, then $\underline{A}_i$ may be assumed constant so that the condition for stability becomes

$$|\lambda_{\underline{A}}| < 1$$

where $\lambda_{\underline{A}}$ are the eigenvalues of $\underline{A} = \underline{A}_i$ = constant. The rate of convergence will also be determined by how close the magnitudes of the eigenvalues are to 1. Since it is rather tedious to evaluate $\underline{A}_i$ and its eigenvalues, as well as to check the basic assumption that $\underline{A}_i$ is in fact constant, the chief value of (6-49) is to show that if $\underline{F}(\underline{\xi}_i)$ and $\underline{G}(\underline{\eta}_i)$ are bounded, there always exists a Δ such that the solution to (6-50) is asymptotically stable, i.e.,

$$\lim_{\eta \to \infty} \underline{\delta}_\eta = 0$$

and hence the iteration called for in the algorithm will be stable for sufficiently small Δ.

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13. ABSTRACT

This report presents an approach to point target tracking based on sequential filtering techniques. The tracking problem is defined in terms of a nonlinear vector differential equation and an appropriate state vector. A Bayesian formulation for the problem is selected which results in a least-squares filter solution. Linearization techniques essential to this approach are incorporated into the development of the solution. A computer program which implements the complete solution algorithm is presented. As part of this computer realization, numerical integration of the equations of motion and numerical evolution of the estimate covariance matrix are discussed in detail.

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